

# Unfolding Polyhedra via Cut-Tree Truncation

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## Abstract

We prove that an infinite class of convex polyhedra, produced by restricted vertex truncations, always unfold without overlap. The class includes the “domes,” providing a simpler proof that these unfold without overlap.

## 1 Introduction

It is a long unresolved question whether or not every convex polyhedron may be cut along edges and unfolded to a single, non-overlapping polygon [DO07]. (Henceforth, we use “unfolding” to mean this type of “edge unfolding.”) An unfolding is determined by a polyhedron  $P$  and a cut-tree  $T$  that spans the vertices of  $P$ . We define a property of a pair  $(P, T)$  that permits derivation of a new pair  $(P', T')$ , where  $P'$  has more vertices than  $P$ , such that  $(P', T')$  determines a non-overlapping unfolding.

We say an unfolding has the *empty sector property* if the circular sector in the unfolding defined by each edge incident to a leaf vertex  $x$  of the cut-tree  $T$  is empty. If  $y$  is the parent of  $x$  in  $T$ , then this sector is defined by  $xy_1$  and  $xy_2$ , where  $y_1$  and  $y_2$  are the two unfolded images of  $y$ , with sector angle the curvature at  $x$ . See Fig. 1(b).

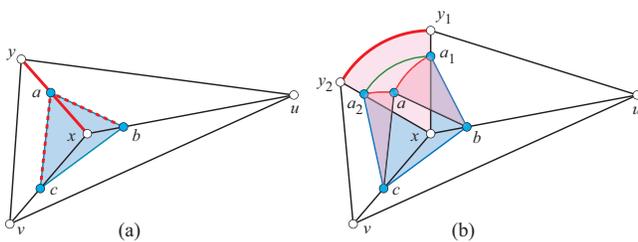


Figure 1: Sector nesting for triangle truncation. (a) Vertex  $x$  of  $P$  is truncated to  $\triangle abc$  in  $P'$ ; (b) The unfolding of  $P$  and of  $P'$  in the vicinity of  $x$ .

Many unfoldings have the empty sector property. For example, take any tetrahedron, and a Y cut-tree, the star from some vertex. This clearly produces an unfolding

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with the empty sector property. Every unfolding of the cube has the empty sector property.

Let  $x$  be a leaf of the cut-tree  $T$ . A *leaf truncation* is a truncation of the vertex  $x$  of the polyhedron  $P$  to  $P'$ , and a corresponding alteration of the cut-tree to  $T'$ , so that, if  $y$  is the parent of  $x$  in  $T$ , and  $(a, p_1, p_2, \dots, p_k)$  is the polygon resulting from the truncation, with  $a \in xy$ , then,  $T'$  follows the “claw”  $(y, a, p_1, p_2, \dots, p_i)$  and  $(y, a, p_k, p_{k-1}, \dots, p_{i+1})$ , leaving some edge  $p_i p_{i+1}$  uncut. For example, if  $x$  has degree 3 in  $P$  (as in Fig. 1(a)), then the truncation polygon is a triangle  $\triangle abc$  ( $bc = p_1 p_2$ ), and the claw becomes the Y  $\{ya, ab, ac\}$ . We call this a *degree-3 leaf truncation*, to which the theorem below is restricted.

## 2 Main Theorem

**Theorem 1** *If a non-overlapping unfolding of a polyhedron  $P$  via a cut-tree  $T$  has the empty sector property, then the cut-tree  $T'$  produced by a degree-3 leaf truncation (a) unfolds  $P'$  without overlap, and (b) has the empty sector property.*

**Proof:** Let  $x$  be a degree-3 leaf of the cut-tree  $T$  with parent  $y$ , and let  $x$  have incident edges  $\{yx, ux, vx\}$ . Let the truncation of  $x$  produce  $\triangle abc$  on the truncated polyhedron  $P'$ , with  $a \in yx$ ,  $b \in ux$ ,  $c \in vx$ , as illustrated in Fig. 1(a). The cut-tree  $T'$  includes the Y  $\{ya, ab, ac\}$ ; the edge  $bc$  is uncut.

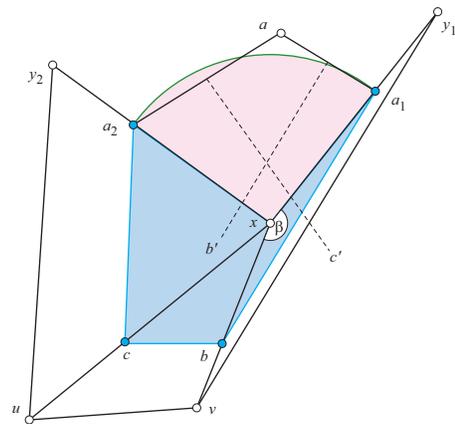


Figure 2: If  $a$  is outside the sector, then the positions of  $b$  and  $c$  are reversed to  $b'$  and  $c'$ .

We now compare the unfolding of  $P$  and of  $P'$  in

the vicinity of  $x$ . The curvature  $\alpha$  at  $x$  is the angle gap in the layout of the triangles  $\triangle xyu$ ,  $\triangle xuv$ ,  $\triangle xvy$  around  $x$  in the unfolding of  $P$ . A key observation is that the unfolding of  $P'$  leaves these triangles fixed in the same position but truncated. The edges  $ub$  and  $vc$  remain uncut, and so maintain the relative positions of the triangles. The truncation triangle  $\triangle abc$  is affixed at edge  $bc$  in the  $P'$  unfolding. (See Fig. 1(b).) Our goal is to prove that  $\triangle abc$  remains inside  $\text{sector}(x, a_1, a_2)$ , and therefore inside the enclosing  $\text{sector}(x, y_1, y_2)$ .

The position of  $a$ , the tip of  $\triangle abc$  in the unfolding of  $P'$ , is the point where the rotations of  $ba_1$  and  $ca_2$  meet. Let  $\beta = \angle a_1xb$  and  $\gamma = \angle a_2xc$ . Then, as is evident in Fig. 2, the rotation of  $a_1$  must start inside  $\text{sector}(x, a_1, a_2)$  in a neighborhood of  $a_1$ , because the convexity of  $\beta$  places  $b$  right of the line through  $xa_1$ . And similarly for  $a_2$ . However, it is conceivable that this rotation ends with  $a$  outside the sector. Suppose it does, as in the figure. Then it must be that  $b$  lies on the perpendicular bisector of  $a_1a$ , because  $a$  is the rotation of  $a_1$  about  $b$  at a distance  $|ba_1|$ ; and similarly  $c$  lies on the perpendicular bisector of  $a_2a$ . But these bisectors must cross inside the sector, each passing to the “wrong side” of  $x$ , because  $a$  is outside the sector (if  $a$  were on the arc  $a_1a_2$ , then the bisector would pass through  $x$ ). This interchanges the presumed positions of  $b$  and  $c$ , a contradiction.

To be more precise on this interchange claim, let  $L$  be the line containing  $bc$ . Then the perpendicular bisector of  $a_1a$  meets  $L$  in a point  $b'$  that is left of the point  $c'$  at which the bisector of  $a_2a$  meets  $L$ . But we labeled the vertices so that  $b$  is right of  $c$  on  $L$ .

Although our figures use  $\alpha < \pi$ , nothing changes with  $\alpha \geq \pi$ . Therefore, we have established the sector nesting property claimed in the lemma.  $\square$

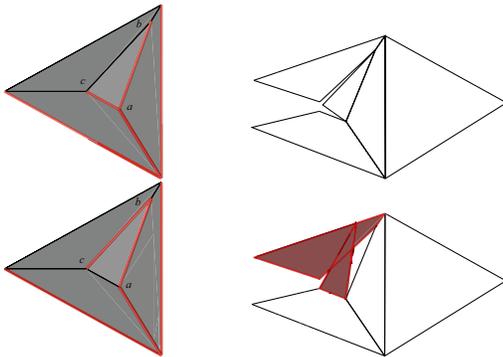


Figure 3: Including the edge  $bc$  from Fig. 1 can lead to overlap.

Fig. 3 shows that the Y-split choice made in the definition of degree-3 truncation is necessary.

**Corollary 2** Any polyhedron  $P$  derived by repeated degree-3 leaf truncations from an initial polyhedron  $P_0$

and cut-tree  $T_0$  with the empty sector property, unfolds without overlap via the derived cut-tree  $T$ .

See Fig. 4 for an example.

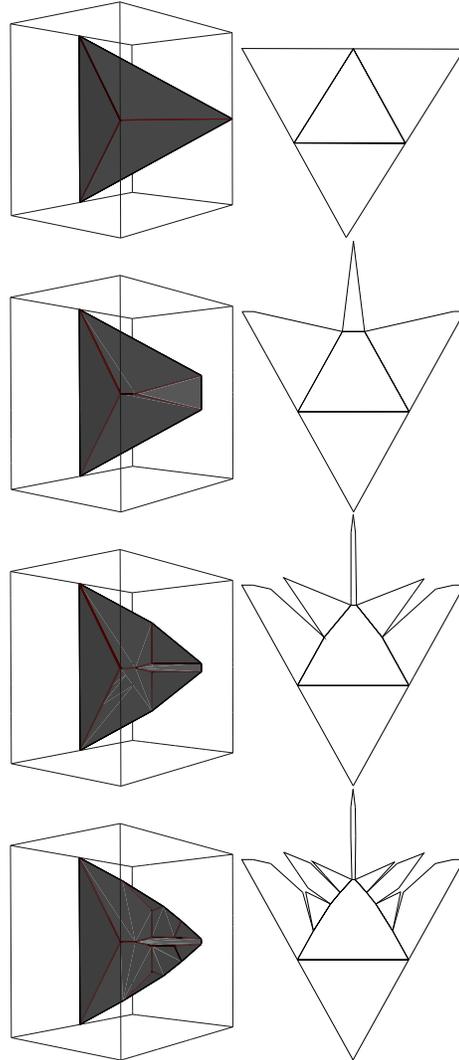


Figure 4: A series of degree-3 truncations refining one corner of a tetrahedron.

### 3 A counterexample for a truncation of degree $> 3$

This theorem does not hold in its most general form for truncation of leaves of degree  $\delta > 3$ . This negative result is demonstrated by a pyramid  $P$  with a five-sided wide and thin base, shown in Fig. 5. The cut tree  $T$  of  $P$  has the empty sector property at every vertex. Note that  $T$  follows the ‘spine’ of  $P$  from base to apex.

We truncate  $P$  to  $P'$ , replacing the apex of  $P$  with a new five-sided face  $f$  (Fig. 6). Retaining all previous cut edges of  $T$  and extending  $T'$  to include four of the new edges of  $f$ , we find that there is no edge of  $f$  at which it may be joined to the unfolding of  $P'$  without conflict

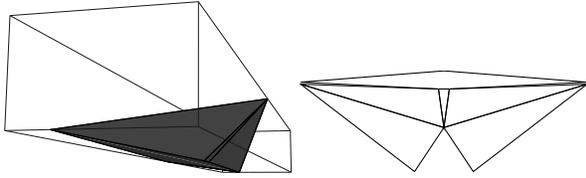


Figure 5: (a) Pyramid  $P$  with five-sided base. (b) The pyramid, unfolded.

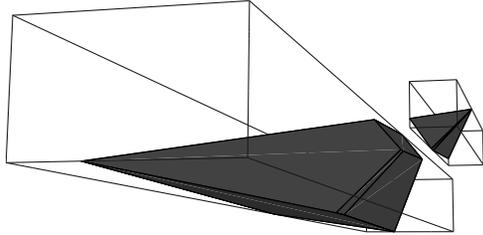


Figure 6: The truncated pyramid  $P'$  with tip removed.

(Fig. 7). This establishes that Theorem 1 cannot be extended to arbitrary  $k$ .

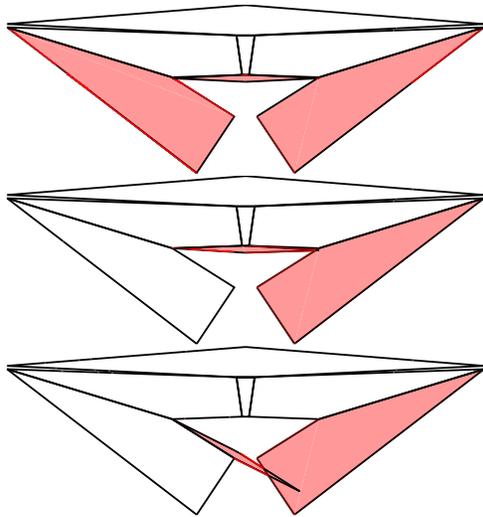


Figure 7: All five attachments of the new face  $f$  to  $P'$  (two symmetric and not shown) lead to overlap.

#### 4 Cutting to achieve degree- $k$ vertices

As shown above, truncating a vertex of degree  $> 3$  may introduce conflict in the unfolding. This prohibits the removal of such vertices through truncation but does not prohibit the creation of higher-degree vertices.

Let  $a$  be a degree-3 leaf node of cut-tree  $T$ , and  $b$  the parent of  $a$ . Perform a leaf-node truncation to produce  $T_\epsilon$ , where the truncation triangle cuts  $ab$  a small distance  $\epsilon$  from  $b$ . By Theorem 1, the unfolding produced by  $T_\epsilon$  strictly avoids overlap. Note that  $b$  is “nearly”

degree-4; see Fig. 8. Letting  $\epsilon \rightarrow 0$  changes  $b$  to truly degree-4 without causing overlap. This process can be repeated on any degree-3 leaf to increase the degree of its immediate parent. Note that this argument results in every  $\epsilon$ -edge being part of  $T$  and therefore cut, which is essential.

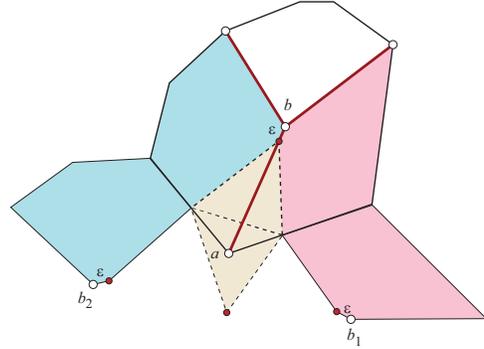


Figure 8: When  $\epsilon \rightarrow 0$ , node  $b$  of  $T_\epsilon$  becomes degree-4.

#### 5 Empty sector property essential

We now show that the empty sector property is necessary for Theorem 1. Consider the convex cap  $C$  in Fig. 9(a), a subset of a larger convex polyhedron not shown.  $C$  is cut by some cut-graph  $T$  to the unfolding shown in Fig. 9(b). This local subset of the unfolding avoids overlap but does not have the empty sector property.

In Fig. 9(c) the circled degree-3 vertex in Fig. 9(a) has been truncated, replacing it with a triangular face and extending  $T$  with new edges. This immediately introduces a conflict into the unfolding, seen in Fig. 9(d).

#### 6 Polyhedra achievable by degree-3 leaf truncation

It is of interest to know which shapes are achievable by degree-3 leaf truncation, for all these shapes are edge-unfoldable. The class depends on the initial  $(P_0, T_0)$  pair. Starting from a pyramid leads to the class of “domes,” which were defined and proved to be unfoldable without overlap in [DO07, Sec. 22.5]. A *dome* is a convex polyhedron with a distinguished *base* polygon  $B$ , and the property that every nonbase face shares an edge with  $B$ .

**Lemma 3** *Starting from a pyramid  $P_0$  and cut-tree  $T_0$  the star of edges incident to the apex  $a$  of  $P_0$ , the polyhedra achievable via degree-3 leaf truncation are all domes.*

**Proof:** Let  $i$  index the  $i$ -th truncation. The leaf nodes of  $T_i$  all lie on  $B_i$ . Degree-3 truncation is defined only on leaf nodes of  $T$ , therefore every truncation will intersect  $B_i$  and create two new vertices which also lie

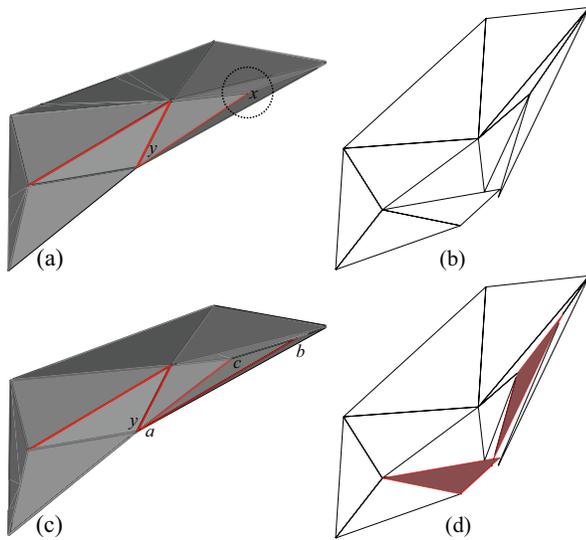


Figure 9: Without the empty sector property, degree-3 vertex truncation cannot guarantee non-overlap.

on  $B_i$ . Thus every face created will have at least two vertices on  $B$  (Cf. Fig. 10), ensuring that  $P_i$  is a dome.  $\square$

Together with the converse, that all domes are achievable via degree-3 truncation, this provides an alternative and simpler proof that domes unfold without overlap. Moreover, Corollary 2 reaches a class of polyhedra larger than domes, for the starting  $P_0$  and  $T_0$  just need the empty sector property. For example, starting from a non-overlapping unfolding of a cube or a dodecahedron will lead to polyhedra that are not domes.

### 7 Conclusion

Although we showed that our result cannot be extended in several possible directions, it may be that it can be extended under certain circumstances. For example, perhaps there are geometric (curvature?) conditions at a leaf vertex of degree  $> 3$  that permit truncations without overlap.

In general, we believe that the notion of deriving a non-overlapping unfolding from a simpler non-overlapping unfolding deserves further study.

### References

[DO07] Erik D. Demaine and Joseph O'Rourke. *Geometric Folding Algorithms: Linkages, Origami, Polyhedra*. Cambridge University Press, June 2007. <http://www.gfalop.org>.

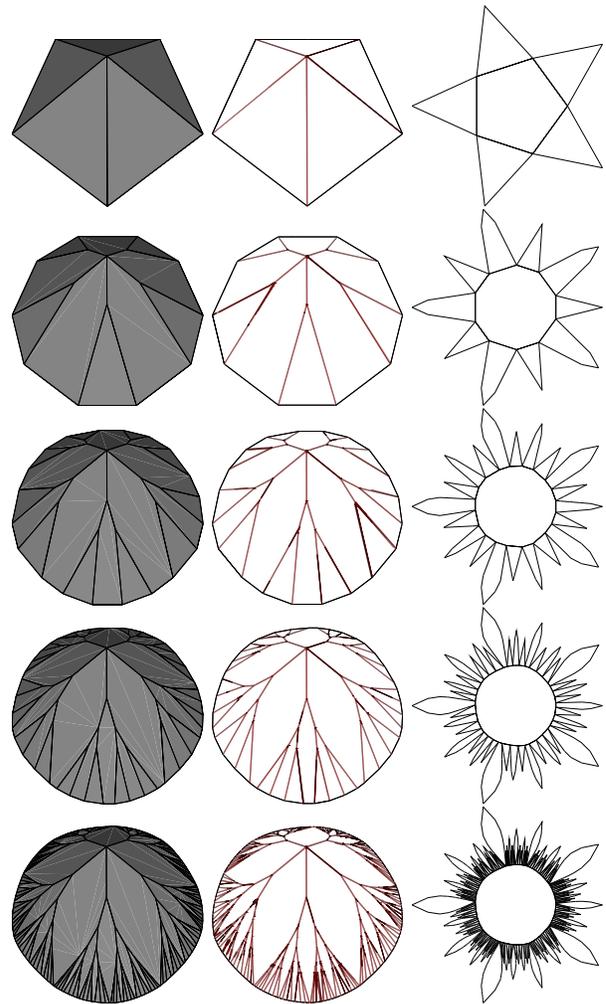


Figure 10: Progressive truncation of  $P_i \rightarrow P_{i+1}$  retains the empty sector property and unfoldability. Left: top view of polyhedron; middle: cut tree; right: unfolding.