Angular Criteria for Unfoldable Convex Polyhedra

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Abstract. The question of whether every convex polyhedron is edge-unfoldable without intersection remains open. In support of efforts towards resolving this question, insights are offered into angular restrictions on the cut-graph of an unfolding of a polyhedral surface which can ensure that the unfolding is without overlap.

§1. Introduction

The following quote is taken from Demaine and O'Rourke's "A Survey of Folding and Unfolding in Computational Geometry" (2005):

> A classic open problem is whether (the surface of) every convex polyhedron can be cut along some of its edges and unfolded into one piece without overlap. [...] It seems folklore that the answer to this question should be yes, but the evidence for a positive answer is actually slim. [DO05]

In [F97] Fukuda showed with a slim tetrahedron that when carefully chosen, even the simplest polyhedral surface could admit a self-intersecting unfolding (Figure 1). This almost trivial example shows that the choice of net in constructing an unfolding can be critical to the unfolding's validity. This was amply demonstrated by O'Rourke in [O98], in which it was shown that as the number of vertices of a convex mesh increases, the percentage of nets which generated invalid, self-intersecting unfoldings goes rapidly to 100%.

However, while these prior works show that many convex meshes exist which admit one or more invalid unfoldings, no example has yet been found of a convex surface which has *no* valid unfolding. The intent of this paper is to offer insights of potential use in the quest for a proof of whether or not all convex polyhedra are edge-unfoldable. Lemmas are

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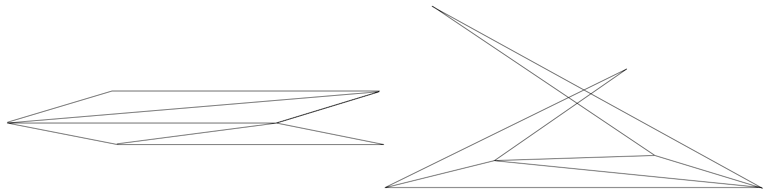

Fig. 1. Two unfoldings of Fukuda and Namiki's slim tetrahedron: without intersection (left) and with intersection (right).

offered which describe angular criteria under which the unfolded net of a polyhedral surface can have no intersection. It is hoped that this will spur ongoing discussion of this fascinating question, and that future work will extend these angular criteria into the robust framework of a proof of convex polyhedral developability.

§2. Definitions

A polyhedral surface is said to be *developable* or *edge-unfoldable* if a subset of its edges can be found which may be cut such that the faces of the mesh remain connected in a net by edges about which the mesh may be unfolded, flattening to the plane without intersection.

The *angle deficit* of a vertex V is 2π minus the sum of the corner angles of the polygonal facets meeting at V ([V94], p.5.) An edge or series of edges broken in the course of developing a model is referred to as a *cut*. Cuts join together to form the *cut-graph*, a connected undirected graph of edges which will be removed to unfold the surface.

A series of edges linked end-to-end in a cut-graph is called a *cut-path*. A cut-path Φ is represented as an ordered vertex list $\{V_0..V_n\}$, at each of which at most two incident edges are cut. Vertex V_0 is called the *root* of the cut. Vertex V_n is called the *tip* of the cut. At the root and tip vertices, only one incident edge is cut.

An edge $\overline{V_k V_{k+1}}$ on the cut-path Φ is congruent to two projected edges in the unfolded net. These duals are labeled $\overline{V_k^L V_{k+1}^L}$ and $\overline{V_k^R V_{k+1}^R}$, where L and R denote the left- and right-hand duals of the unfolding edge respectively (Figure 2). The set of all left-hand edges projected from Φ is labeled Φ^L. The set of all right-hand edges projected from Φ is labeled Φ^R.

The root of a cut is cut by only a single edge, and so projects to only a single point in the unfolding plane. To distinguish the projection of V_0 from the original on the source surface, the projection is labeled V_0^O. This notation is chosen to emphasize that V_0^O denotes, in some sense, the 'origin' in the plane of the unfolding on this cut-path.

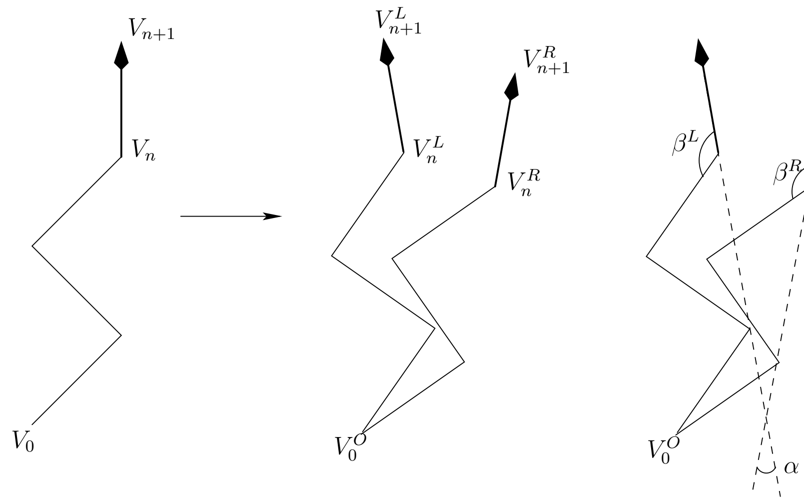

Fig. 2. Unfolding projects the vertices of the cut-path Φ to two sets of coordinates in $\Re 2$, Φ^L and Φ^R. $\alpha = AD(\Phi) = \sum_{i=0}^{n} AD(V_i)$. $\beta = min(\beta^L, \beta^R)$.

In the interest of brevity in notation, the symbol α_n is adopted for the angle between the n^{th} pair of unfolded edges of a cut-path Φ. The symbol β_n identifies the angle from V_0^O to the nearer of the two unfolded projections of $\overline{V_n V_{n+1}}$, measured at V_n (Figure 2):

$$\alpha_n = \angle(\overline{V_n^L V_{n+1}^L}, \overline{V_n^R V_{n+1}^R}) = \sum_{i=0}^{n} AD(V_i),$$
$$\beta_n = min(\angle(V_0^O V_n^L V_{n+1}^L), \angle(V_0^O V_n^R V_{n+1}^R)).$$

The total angle deficit of a convex polyhedron is 4π[W99]. This is then the maximum possible value of α over the longest possible cut-path on the surface, visiting every vertex in the surface once.

§3. Angular Restrictions to Ensure Developability

3.1. Preventing overlap between each pair of unfolded edges

Lemma 1. *Given a cut-path $\Phi = \{V_0..V_n\}$ where $\left\|\overline{V_0^O V_n^L}\right\| = \left\|\overline{V_0^O V_n^R}\right\|$, if $\beta_n \geq \pi/2$ then it is impossible for $\overline{V_n^L V_{n+1}^L}$ to intersect $\overline{V_n^R V_{n+1}^R}$.*

Proof: An affine transformation may be constructed which maps V_0^O to the origin of $\Re 2$, V_n^L to [0,1] and V_{n+1}^L, V_n^R and V_{n+1}^R to unique coordinates in positive X on the XY plane. These points are labeled O, A, B, C and D, respectively. Note that attention is restricted to the range $0 \leq \alpha \leq \pi$.

Expressing C and D in terms of A and B rotated about the origin by α radians and solving the system of linear equations $\overline{AB}(t) = A + t(B - A)$, $\overline{CD}(u) = C + u(D - C)$ for t, u, yields

$$t = \frac{1 - B_y + B_x tan(\alpha/2)}{B_x^2 + (B_y - 1)^2},$$
$$u = \frac{1 - B_y + B_x(cot(\alpha) - csc(\alpha))}{B_x^2 + (B_y - 1)^2}.$$

Rewriting B as $[k * cos(\beta), 1 + k * sin(\beta)]$ gives

$$t = (cos(\beta)tan(\alpha/2) - sin(\beta))/k$$
$$u = -(cos(\beta)tan(\alpha/2) + sin(\beta))/k$$

If $0 \leq \alpha \leq \pi$ then $tan(\alpha/2)$ is positive. If $\pi/2 \leq \beta \leq \pi$ then $cos(\beta)$ is negative and $sin(\beta)$ is positive. If $0 \leq \alpha \leq \pi$ and $\pi/2 \leq \beta \leq \pi$, then $t \leq 0$, i.e., the two line segments cannot intersect. □

3.2. Preventing overlap between an edge and previous edges on the cut-path

Lemma 2. *Given a cut-path $\Phi = \{V_0..V_{n+1}\}$ where V_n^L and V_n^R have unfolded to the same distance d in the plane from V_0^O, if $\beta_n \geq \pi/2$ then it is impossible for $\overline{V_n^L V_{n+1}^L}$ to intersect any edge before n in Φ^R. Likewise, $\overline{V_n^R V_{n+1}^R}$ cannot cross any prior edge in Φ^L.*

Proof: Recalling that β_n is the minimum of the two angles $\angle(V_0^O V_n^L V_{n+1}^L)$ and $\angle(V_0^O V_n^R V_{n+1}^R)$, $\beta \geq \pi/2$ implies that both $\left\|\overline{V_0^O V_{n+1}^L}\right\|$ and $\left\|\overline{V_0^O V_{n+1}^R}\right\|$ are greater than d. Both projections of V_{n+1} will lie outside the circle of radius d centered on V_0^O. Geometry permitting, each pair of unfolded edges in a cut-path may be chosen to lie between circles of progressively larger radius centered on V_0^O, and so will never intersect the unfolded projections of any earlier edge. □

3.3. Extending a cut-path while ensuring developability

Lemma 3. *Given a cut-path $\Phi = \{V_0..V_n\}$ on a closed convex surface there must exist at least one vertex V_{n+1} in the one-ring of V_n such that $\beta_n \geq \pi/2$.*

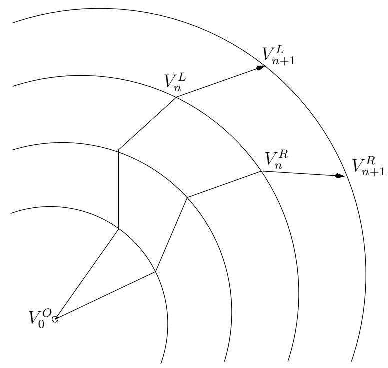

Fig. 3. Nested circles bound regions of intersection.

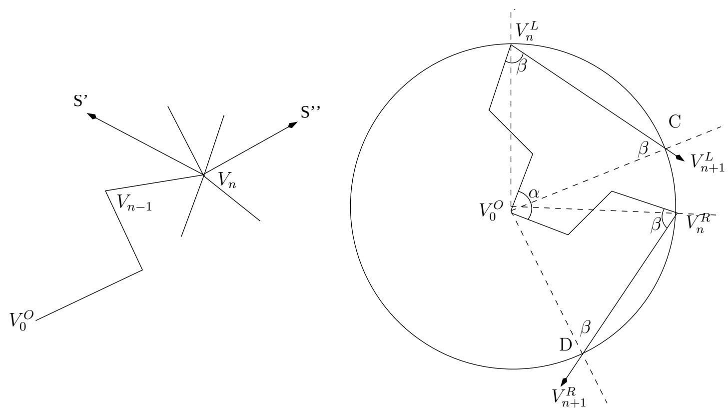

Fig. 4. (a) Extending a cut-path without introducing the possibility of intersection. (b) Sufficiently large angle deficit eliminates the possibility of intersection.

Proof: For every vertex S' adjacent to V_n (Figure 4a), on a mesh of convex facets there must exist at least one other vertex S" such that $\pi/2 \leq \angle(S'V_nS'') \leq 3\pi/2$. Thus if it were the case that selecting S' as V_{n+1} gave a value for β_n which was less that $\pi/2$, then selecting S" as V_{n+1} instead would give $\pi/2 \leq \beta_{S'} + \angle$(S'V$_0V_n$S")$\leq 3\pi/2$. Therefore selecting S" as V_{n+1} is guaranteed to yield $\beta_n \geq \pi/2$. □

Lemma 4. *Given a cut-path $\Phi = \{V_0..V_{n+1}\}$ where V_n^L and V_n^R have unfolded to the same distance d in the plane from V_0^O, if $\beta_n > (\pi - \alpha)/2$ then it is impossible for $\overline{V_n^L V_{n+1}^L}$ to intersect $\overline{V_n^R V_{n+1}^R}$.*

Fig. 5. Unfolding the sphere by cutting edges along geodesic curves.

Proof: If $\beta_n \geq \pi/2$ then there is no intersection (Lemma 1b). If $\beta_n < \pi/2$ then the edge $\overline{V_n^L V_{n+1}^L}$ intersects the circle of radius d at some point C (Figure 4a). The triangle $V_n^L C V_0^O$ is an isosceles triangle with angles β_n, β_n, $\pi - 2\beta_n$. $\alpha_n > \pi - 2\beta_n$ implies that V_n^R must lie outside the triangle $V_n^L C V_0^O$, indicating that the triangle $V_n^R D V_0^O$ does not overlap $V_n^L C V_0^O$ at any point other than V_0^O. Therefore if $\beta_n > (\pi - \alpha_n)/2$ then $\overline{V_n^L V_{n+1}^L}$ and $\overline{V_n^R V_{n+1}^R}$ cannot intersect. □

§4. Conclusions and Future Directions

Taken together, these four lemmas may be informally summarized as, "if a cut doesn't bend too far onto itself before it's opened wide enough, its two sides are sure to never cross." Slightly more formally, this could be phrased as, "cuts should travel along geodesic lines." This inspires a rough sketch of a final proof: a single vertex, chosen arbitrarily, would become the root of a tree of cut-paths. It should be possible to show through an extension of Lemma 3 that every vertex on the surface could be connected by a cut-path to this seed vertex, and that those cut-paths could be constrained to travel within some tolerance of the geodesic linking their targets to the root point. Proximity to the geodesic would allow Lemmas 1 and 2 to prove a lack of conflict for the early portions of the path (where the total angle deficit had not yet exceeded π) and Lemma 4 would extend that support for greater values of α. The resulting cut-graph would look like a star (Figure 5), similar in overall construction to Agarwal et al's Star Unfolding [AAOS97].

In order to realize the proof, a great deal of future work remains. The constraint of matching the unfolded radii of V_n^L and V_n^R, an essential part of each proof, is particularly difficult to meet in that it places strong restrictions on the set of admissible surfaces and cut-paths. While this is not an issue for the unfolding of, for example, a parametric model of the globe along longitudinal lines, very few models are so well-behaved or offer edges which can be so cleanly mapped to geodesics; a more general form is essential. The mathematics given also address only α and β in the range $\{0..\pi, (\pi - \alpha)/2..\pi\}$; full coverage of the configuration space is required.

This is only a rough sketch of the possible final structure of the proof. Nonetheless, if it proves possible to extend these lemmas to form a robust framework, then there will be a real hope of achieving a proof of the unfoldability of convex polyhedra. At the same time, if an undevelopable polyhedron does exist, then a good chance of finding it lies in finding the counterexamples to the approaches described.

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